GENERALIZED STRONG CURVATURE SINGULARITIES AND COSMIC CENSORSHIP

WIESŁAW RUDNICKI[*]

Institute of Physics, University of Rzeszów, Rejtana 16A, 35-959 Rzeszów, Poland

ROBERT J. BUDZYŃSKI[†]

Department of Physics, Warsaw University, Hoża 69, 00-681 Warsaw, Poland

WITOLD KONDRACKI[‡]

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-950 Warsaw, Poland

Abstract

A new definition of a strong curvature singularity is proposed. This definition is motivated by the definitions given by Tipler and Królak, but is significantly different and more general. All causal geodesics terminating at these new singularities, which we call *generalized strong curvature singularities,* are classified into three possible types; the classification is based on certain relations between the causal structure and the curvature strength of the singularities. A cosmic censorship theorem is formulated and proved which shows that only one class of generalized strong curvature singularities, corresponding to a single type of geodesics according to our classification, can be naked. Implications of this result for the cosmic censorship hypothesis are indicated.

Keywords: Spacetime singularities; Cosmic censorship; Causal structure; Geodesics

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[*]E-mail: rudnicki@atena.univ.rzeszow.pl

[†]E-mail: Robert.Budzynski@fuw.edu.pl

[‡]E-mail: witekkon@panim.impan.gov.pl

1   Introduction

The cosmic censorship hypothesis (CCH) of Penrose [1, 2] says that in generic situations, all spacetime singularities arising from regular initial data are always hidden behind event horizons and hence invisible to outside observers (no naked singularities). This hypothesis plays a fundamental role in the theory of black holes and has been recognized as one of the most important open problems in classical general relativity. There exist many exact solutions of Einstein's equations which admit naked singularities. However, Penrose [2] has stressed that the exact solutions with special symmetries have a rather limited value for verification of the CCH and what is required here is an understanding of the generic case.

One possible approach to this problem is to propose a class of generic singularities and then attempt to formulate and prove a censorship theorem which would constrain or prohibit the occurrence of naked singularities of the proposed class. Tipler *et al.* [3] and Królak [4] have argued that all singularities arising in generic situations should be of the strong curvature type. These singularities have the property that all objects approaching them are crushed to zero volume. The idea of a strong curvature singularity was introduced by Ellis and Schmidt [5] and defined in precise geometrical terms by Tipler [6] (a slightly different definition was given by Królak [4], see below). Królak [4, 7] formulated and proved some censorship theorems that ruled out a class of naked singularities of the strong curvature type. Unfortunately, these results rely heavily on a further assumption (the so-called simplicity condition) which need not hold for generic spacetimes.

It should be stressed that there is no hope for finding a proof of the CCH using only the assumption that all singularities occurring in a given spacetime are of the strong curvature type. This follows just from the fact that naked singularities of this type *do* occur in certain exact solutions of Einstein's equations. For instance, they occur in the Tolman-Bondi solution representing spherically symmetric inhomogeneous collapse of dust (see, e.g., Ref. [8]; see also Ref. [9] and references therein). Such naked singularities also occur in more general models of dust collapse — namely, in the Szekeres spacetimes which do not have any Killing vectors [10, 11]. Unnikrishnan [12] has argued that the existence of naked strong curvature singularities in the Tolman-Bondi solution can be ruled out by imposing certain reasonable constraints on the initial distribution of the energy density of dust. This argument, however, depends crucially on the spherical symmetry of the solution, and so cannot be applied to naked singularities occurring in more general cases — e.g. in the Szekeres spacetimes. It is worth recalling here that Bonnor [13] remarked, in another context, that "the Szekeres solution has a good deal of symmetry, even though it has in general no Killing vectors." It is thus possible that the existence of naked singularities of the strong curvature type will always be accompanied by spacetime symmetries and/or instabilities of some sort, and so one can still hope to prove a formulation of the CCH involving — besides the assumption that all singularities are of strong curvature — a suitable criterion of genericity or stability.

To help identify such a criterion, it may be useful to establish and study various relations between strong curvature singularities and the causal structure in their neighborhood — such relations get to the heart of cosmic censorship. For this purpose, we need a new definition of a strong curvature singularity — one that not only describes the curvature strength of the singularity, but additionally enables one to relate the strength with

properties of the causal structure in a neighborhood of the singularity. In Section II of this paper, we shall propose such a definition. Our definition is motivated by the definition of a strong curvature singularity given by Tipler and its modifications by Królak, but is significantly different and more general. The difference is in that our definition involves a certain focusing condition on solutions of the Raychaudhuri equation not only along a *single* causal geodesic, as in Tipler's and Królak's case, but rather along *all* causal geodesics in some small neighborhood about a given geodesic that reaches the singularity. All causal geodesics terminating at the singularities described by our definition, which we will refer to as *generalized strong curvature singularities*, will next be classified into three possible types; the classification is based on certain relations between the causal structure and the curvature strength of the singularities. Further on, in Sections III and IV, we shall formulate and prove a cosmic censorship theorem which shows that only one class of generalized strong curvature singularities, corresponding to a single type of geodesics according to our classification, can be naked.

This paper was inspired by ideas given in Ref. [14]. Some of the results presented here are refinements of those announced without proofs in Ref. [15]. The notation and fundamental definitions are as those in the monograph of Hawking and Ellis [16].

2 Generalized Strong Curvature Singularities

Before we give our definition, we need to recall one standard result on the behavior of geodesic congruences. Let $\lambda(t)$ be an affinely parametrized null (resp. timelike) geodesic. A congruence of null (timelike) geodesics infinitesimally neighboring $\lambda(t)$ and originating from a point on $\lambda(t)$ is characterized by two parameters: the expansion θ and the shear σ. The rate of change of the expansion θ along $\lambda(t)$ is given by the Raychaudhuri equation:

$$\frac{d\theta}{dt} = -R_{ab}K^a K^b - 2\sigma^2 - \frac{1}{n}\theta^2, \tag{1}$$

where R_{ab} is the Ricci tensor, K^a is the tangent vector to $\lambda(t)$, $n = 2$ if $\lambda(t)$ is a null geodesic and $n = 3$ in the case when it is timelike (see pp. 78–88 of Ref. [16]).

Let us also recall the definition of a strong curvature singularity introduced by Królak [4].

Definition 1: *Let λ be a future-endless, future-incomplete null (timelike) geodesic. λ is said to terminate in the future at a strong curvature singularity if, for each point $p \in \lambda$, the expansion θ of every future-directed congruence of null (timelike) geodesics emanating from p and containing λ becomes negative somewhere on λ.*

This definition is equivalent to condition K given in Ref. [19]. Królak's definition generalizes Tipler's definition of a strong curvature singularity because it implies weaker restrictions on the divergence of the curvature near the singularity [19].

We can now introduce our definition of a generalized strong curvature singularity.

Definition 2: *Let λ be a future-endless, future-incomplete null (timelike) geodesic. We say that λ terminates in the future at a generalized strong curvature singularity if, for each sequence of endless null (timelike) geodesics, $\{\lambda_n\}$, for which λ is a limit curve, at least one of the following conditions holds:*

(i) *for each point $p \in \lambda$ and for each neighborhood N of p there exist a geodesic $\widetilde{\lambda} \in \{\lambda_n\}$ and a point $\widetilde{q} \in \widetilde{\lambda} \cap N$ such that the expansion θ of a future-directed congruence of null (timelike) geodesics emanating from $\widetilde{q}$ and containing $\widetilde{\lambda}$ becomes negative somewhere on $\widetilde{\lambda}$;*

(ii) $I^-(\lambda) = I^-(\lambda_n)$ *for almost all geodesics belonging to $\{\lambda_n\}$.*

Analogously, λ is said to terminate in the past at a generalized strong curvature singularity if the time-reverse versions of the above conditions hold for λ.

Remark: By the term *limit curve* we mean in the above a curve that satisfies the definition given on p. 185 of Ref. [16].

It is easy to notice that every causal geodesic that terminates at a strong curvature singularity as defined by Królak also terminates at a generalized strong curvature singularity as defined above. To see this, consider, e.g., a null geodesic λ that terminates in the future at Królak's singularity. Next, take any sequence of endless null geodesics, $\{\lambda_n\}$, for which λ is a limit curve. Let p be an arbitrary point on λ, and let $\{p_n\}$ be an arbitrary sequence of points converging to p such that, for each n, $p_n \in \lambda_n$. In accordance with Królak's definition, the expansion θ of every future-directed congruence of null geodesics outgoing from p and containing λ must become negative. It follows by continuity that there must exist certain points $p_k \in \{p_n\}$ sufficiently close to p such that, for each k, the expansion θ of a future-directed congruence of null geodesics outgoing from p_k and containing $\lambda_k \in \{\lambda_n\}$ shall become negative as well. This clearly means that $\{\lambda_n\}$ shall obey condition (i) of Definition 2.

Let us now turn to condition (ii) of Definition 2. This condition has no direct equivalent in either Tipler's or Królak's definition; however, it appeals to the original idea behind the concept of a strong curvature singularity [5]. To see this, observe first that the requirement $I^-(\lambda) = I^-(\lambda_n)$ found in condition (ii) means that both λ and λ_n must reach exactly the same point in the *c-boundary* of the spacetime (p. 218 of Ref. [16]). The geodesics λ_n represent trajectories of point particles in motion towards the singularity within some small neighborhood about λ. Following the original idea of a strong curvature singularity, all small physical objects should be crushed to zero volume as they approach the singularity. Condition (ii) corresponds to the particular case where they are crushed to a single point in the *c*-boundary.

It is worth mentioning here that the *c*-boundary of cosmological models has been investigated by Tipler [18] in the context of limitations imposed on computation by general relativity. He has shown that a true universal Turing machine can be constructed only in a closed universe whose final singularity is a single point in the *c*-boundary topology. Thus all causal geodesics reaching this singularity will satisfy condition (ii) of Definition 2 and hence terminate in our generalized strong curvature singularity; moreover, they will be of type A according to the classification given below.

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The following definition provides a complete classification of causal geodesics terminating at the singularities introduced in Definition 2.

Definition 3: *Let λ be a future-endless null (timelike) geodesic terminating in the future at a generalized strong curvature singularity, and let $\{\lambda_n\}$ be a sequence of endless null (timelike) geodesics for which λ is a limit curve. λ is said to be of type:*

- *A, if condition (2) of Definition 2 holds for each $\{\lambda_n\}$;*

- *B, if, for each $\{\lambda_n\}$ which does not satisfy condition (2) of Definition 2, there exist a geodesic $\widetilde{\lambda} \in \{\lambda_n\}$ and a point $\widetilde{q} \in \widetilde{\lambda} - I^-(\lambda)$ such that the expansion θ of a future-directed congruence of null (timelike) geodesics emanating from $\widetilde{q}$ and containing $\widetilde{\lambda}$ becomes negative somewhere on $\widetilde{\lambda}$;*

- *C, if λ is neither of type A nor B.*

Remark: If the λ mentioned above is a null geodesic that admits a segment contained in the boundary, $\dot{I}^-(\lambda)$, of its chronological past, one can always find a sequence of endless null geodesics, $\{\lambda_n\}$, for which λ is a limit curve, such that none of the λ_n will be contained in the closure of $I^-(\lambda)$. Clearly, such a sequence cannot satisfy condition (ii) of Definition 2, and so λ cannot be of type A. In general, however, there may exist a future-endless null geodesic λ that never intersects the boundary of its chronological past. For example, if λ terminates in the future at a curvature singularity and the curvature diverges fast enough along λ, every future-endless segment of λ may admit a pair of conjugates points, which implies that no segment of λ can be contained in the achronal boundary $\dot{I}^-(\lambda)$. In this case λ may be of type A.

3   Censorship Theorem

For the sake of simplicity, we shall restrict our considerations to *weakly asymptotically simple and empty* (WASE) spacetimes (p. 225 of Ref. [16]). Such a spacetime (M, g) can be conformally imbedded in a larger spacetime $(\widetilde{M}, \widetilde{g})$ as a manifold with boundary $\overline{M} = M \cup \partial M$, where the boundary ∂M consists of two null surfaces $\mathcal{J}^+$ and $\mathcal{J}^-$ that represent future and past null infinity, respectively. Moreover, there exists an open neighborhood U of ∂M in $\widetilde{M}$ such that $U \cap M$ coincides with part of an asymptotically simple and empty spacetime (M', g'), which means that all possible singularities of (M, g) can only occur in the region $M - U$. Since the CCH is concerned with singularities that develop from an initially non-singular state, we shall deal here only with such WASE spacetimes (M, g) in which the region $M - U$ does not extend arbitrarily far into the past and to a spatial infinity.[a] To make this precise, we shall assume that (M, g) admits a partial Cauchy surface S for which the following two conditions hold:

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[a]Due to its generality, the definition of a WASE spacetime given in Ref. [16] has the unwanted feature that the region $M - U$ might extend to a spatial infinity. For a deeper discussion of this problem, see Ref. [19].

(i) *S has an asymptotically simple past* (p. 316 of Ref. [16]);

(ii) *every null geodesic μ generating $\mathcal{J}^+$ admits a past-endless segment μ_0 such that $\mu_0 \subset I^+(S, \overline{M})$ and*

$$I^-(\mu_0, \overline{M}) \cap I^+(S) \subset D^+(S) \cap U.$$

Such a surface S will be called a *regular* partial Cauchy surface. (This definition is a slight modification of that previously used in Ref. [20].)

Let us now recall that a WASE spacetime (M, g) is said to be *future asymptotically predictable* from a partial Cauchy surface S if the future null infinity $\mathcal{J}^+$ is contained in the closure of the future Cauchy development $D^+(S, \overline{M})$ (p. 310 of Ref. [16]). Future asymptotic predictability is a mathematically precise statement of cosmic censorship for (M, g), since it ensures that there will be no singularities to the future of S which are naked, i.e. which are visible from $\mathcal{J}^+$.

We are now in a position to state our cosmic censorship theorem. By this theorem, only singularities corresponding to null geodesics of type C can be naked.

Theorem 1: *Let (M, g) be a WASE spacetime admitting a regular partial Cauchy surface S. Suppose furthermore that the following conditions hold:*

(i) *the null convergence condition, i.e. $R_{ab} K^a K^b \geq 0$ for every null vector K^a of (M, g);*

(ii) *the generic condition, i.e. every endless null geodesic of (M, g) admits a point at which $K_{[a} R_{b]cd[e} K_{f]} K^c K^d \neq 0$, where K^a is the tangent vector to the geodesic;*

(iii) *(M, g) admits no naked points-at-infinity, i.e. for each point $p \in \mathcal{J}^+$, every future-endless, future-complete null geodesic of (M, g) contained in $\bar{I}^-(p, \overline{M}) \cap D(S)$ has a future endpoint on $\mathcal{J}^+$ in $\overline{M}$;*

(iv) *every incomplete null geodesic of (M, g) terminates at a generalized strong curvature singularity.*

If (M, g) is not future asymptotically predictable from S, then there must exist a null geodesic $\lambda \subset \operatorname{int} D(S)$ which terminates in the future at a generalized strong curvature singularity and is of type C.

Remarks: Conditions (i) and (ii) of Theorem 1 are reasonable requirements for *any* physically realistic model of a classical spacetime; they have been discussed extensively in the literature on the singularity theorems (see, e.g., Refs. [3, 16]).

Condition (iii) is a slightly weakened version of that previously used by Newman and Joshi [21] in their censorship theorem. This condition ensures that any possible breakdown of future asymptotic predictability in (M, g) must be associated with the occurrence of a naked singularity and not of a naked *"point-at-infinity"*. Known examples of WASE spacetimes with naked singularities contain no naked points-at-infinity in the sense of condition (iii). It should, however, be stressed that this condition restricts to some extent the generality of our result, since it is conceivable that some spacetimes might contain both naked singularities and naked points-at-infinity.

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Note also that no causality conditions, other than those implied by the existence of S, are imposed on (M, g) in Theorem 1. Thus this theorem may be applied to naked singularities associated with causality violation. The possible existence of such naked singularities results from Tipler's singularity theorem [22].

4   Proof of the Theorem

Now we shall prove our theorem; the following two lemmas serve this purpose.

Lemma 1: *Under the assumptions of Theorem 1, suppose that (M, g) is not future asymptotically predictable from S. Then there must exist a past-incomplete null geodesic $\eta \subset H^+(S, \overline{M})$ which has a future endpoint $p \in \mathcal{J}^+$. Moreover, the following conditions hold:*

(a) *there exists a point $r \in \eta \cap M$ such that the closure of $I^-(r) \cap S$ is compact;*

(b) *for each point $c_i \in I^-(p, \overline{M}) \cap D^+(S)$, there exists a null geodesic $\lambda_i \subset \dot{I}^+(c_i, \overline{M})$ which extends from c_i to some point on $\mathcal{J}^+$.*

Proof: Let us first observe that from condition (ii) of the definition of S it follows that every generator of $\mathcal{J}^+$ must intersect the closure of $D^+(S, \overline{M})$. In addition, as S has an asymptotically simple past by condition (i), $\mathcal{J}^-$ must be contained in the closure of $D^-(S, \overline{M})$. This implies that (M, g) is *partially* asymptotically predictable from S as defined by Tipler [22]. Thus, as (M, g) is not future asymptotically predictable from S, by Proposition 2 of Ref. [22] there must exist a past-endless null geodesic generator η of $H^+(S)$ with future endpoint $p \in \mathcal{J}^+$. Following steps as in the proof of Theorem 1 of Ref. [22], we can now easily show, by making use of conditions (i) and (ii) of our theorem, that η must be incomplete in the past.

We shall now show that condition (a) holds. Let us choose an arbitrary point $r \in \eta \cap M$; we shall demonstrate that the set $R \equiv \overline{I^-(r) \cap S}$ is compact. Since S has an asymptotically simple past, the proof of this fact will be quite similar to the proof of the Theorem of Ref. [23].

Assume, to the contrary, that R is not compact. Then, by Lemma 2 of Ref. [23], there must exist a sequence of null geodesics η_i with future endpoints r_i converging to r, such that each η_i will be a generator of the achronal boundary $\dot{I}^-(r_i)$ and will have a past endpoint on $\mathcal{J}^-$; moreover, the geodesic η must be a limit curve of the sequence $\{\eta_i\}$. As η is past-incomplete, by condition (iv) of Theorem 1, it must terminate in the past at a generalized strong curvature singularity. Thus $\{\eta_i\}$ must fulfill at least one of the time-reverse versions of conditions (i) and (ii) of Definition 2. Since each of the η_i has a past endpoint on $\mathcal{J}^-$, for each η_i one has $I^+(\eta_i) \neq I^+(\eta)$, as $\eta \subset I^+(S)$ while $\mathcal{J}^- \subset I^-(S, \overline{M})$. It is thus clear that the time-reverse version of condition (ii) cannot be fulfilled, and so the time-reverse version of condition (i) must be. There must thus exist a geodesic $\widetilde{\eta} \in \{\eta_i\}$ and a point $\widetilde{q} \in \widetilde{\eta}$ such that the expansion θ of a past-directed null geodesic congruence emanating from $\widetilde{q}$ and containing $\widetilde{\eta}$ must become negative somewhere on $\widetilde{\eta}$. In this case, by Proposition 4.4.4 of Ref. [16] and condition (i) of our Theorem 1, there would exist a point conjugate to $\widetilde{q}$ along $\widetilde{\eta}$, since $\widetilde{\eta}$ is

past-complete as it has a past endpoint on $\mathcal{J}^-$. However, by Proposition 4.5.12 of Ref. [16], the existence of a pair of conjugate points on $\widetilde{\eta}$ would contradict the achronality of $\widetilde{\eta}$. In view of this contradiction we must conclude that R is compact.

We shall now show that condition (b) holds. Let c_i be an arbitrary point in $I^-(p, \overline{M}) \cap D^+(S)$, and let μ be a generator of $\mathcal{J}^+$ that passes through p. We shall first show that μ must leave $I^+(c_i, \overline{M})$. To see this, suppose that μ were contained in $I^+(c_i, \overline{M})$. Then the past set $X \equiv \bigcap_{a \in \mu_0}[I^-(a, \overline{M}) \cap M]$, where μ_0 denotes the past-endless segment of μ mentioned in condition (ii) of the definition of S, would be non-empty, as it would contain c_i. From the definition of X it follows that any null geodesic generator of the boundary of X cannot have its future endpoint on $\mathcal{J}^+$; otherwise, such a point would have to coincide with a past endpoint of μ_0, which is impossible as μ_0 is past-endless. But from the definition of the surface S it follows that X would have to be contained in an open neighborhood U of $\mathcal{J}^+ \cup \mathcal{J}^-$, such that $U \cap M$ coincides with part of an asymptotically simple and empty spacetime, which implies that all null geodesics generating the boundary of X would have to have their future endpoints on $\mathcal{J}^+$. This contradiction shows that μ must leave $I^+(c_i, \overline{M})$, so there must exist a point $b \in \mu \cap \dot{I}^+(c_i, \overline{M})$. Since $b \in \mu \cap J^-(p, \overline{M})$, and $p \in H^+(S, \overline{M})$, it follows that b must belong to the closure of $D^+(S, \overline{M})$. Therefore all the past-directed null geodesics outgoing from b, with the exception of μ, must enter the interior of $D^+(S)$ immediately after leaving b; otherwise they would be generators of $\mathcal{J}^+$, which is impossible. Thus, as $\text{int} D^+(S)$ is a causally simple set, there must exist a null geodesic $\lambda_i \subset \dot{I}^+(c_i, \overline{M})$ which extends from c_i to $b \in \mathcal{J}^+$, as required in condition (b). $\qquad \square$

Lemma 2: *Let r be a point on $H^+(S)$, where S is a partial Cauchy surface. If the set $\dot{I}^-(r) \cap J^+(S)$ is compact, then every null geodesic generator of $H^+(S)$ through r is geodesically complete in the past direction.*

Proof: The proof of this lemma is identical to the proof of Lemma 8.5.5 of Ref. [16]. In the course of the proof we only need to consider the set $\dot{I}^-(r) \cap J^+(S)$ instead of the Cauchy horizon $H^+(S)$. $\qquad \square$

Proof of Theorem 1: Assume that (M, g) is not future asymptotically predictable from S. Then, by Lemma 1, there exists a past-incomplete null geodesic $\eta \subset H^+(S)$ which has a future endpoint $p \in \mathcal{J}$; moreover, there exists a point $r \in \eta \cap M$ such that $R \equiv \overline{I^-(r) \cap S}$ is compact. As η is past-incomplete, by Lemma 2 the set $Q \equiv \dot{I}^-(r) \cap J^+(S)$ cannot be compact. Let us put a timelike vector field on M (such a field will always exist since M admits a Lorentz metric g). If every integral curve of this field that intersects the set R also intersects the set Q, we would have a continuous one-to-one mapping of R onto Q, and hence Q would have to be compact. Therefore there must exist a future-endless timelike curve $\alpha \subset I^-(r)$.

Let $\mathcal{P}$ be the family of all sets of the form $I^-(\alpha)$, where α is a future-endless timelike curve contained in $I^-(r)$; and let $\widehat{P}$ be a maximal chain determined in $\mathcal{P}$ by the relation of inclusion. Denote now by P_0 the set $\bigcap\{P : P \in \widehat{P}\}$. This set is non-empty. To see this, note first that no member of $\widehat{P}$ can be contained in $I^-(S)$;

otherwise, there would exist a future-endless timelike curve $\alpha \subset I^-(S)$, which is impossible as $I^-(S) = D^-(S)$. Note now that, for each $P \in \widehat{P}$, the set $\overline{P \cap S}$ must be compact, as it is a closed subset of the compact set R. This clearly implies, by the definition of P_0, that the set $A \equiv \overline{P_0 \cap S}$ is non-empty and compact. In addition, from the definition of P_0 it follows that there must exist a future-endless timelike curve α_0 such that $P_0 = I^-(\alpha_0)$. Note also that P_0 must be a minimal element of $\widehat{P}$, i.e. we will have $P_0 = I^-(\beta)$ for every future-endless timelike curve $\beta \subset P_0$.

Let us now fix some future-endless timelike curve α_0 such that $P_0 = I^-(\alpha_0)$. Let $\{c_i\}$ be a sequence of points on $\alpha_0 \cap D^+(S)$ such that, for each i, $c_{i+1} \in I^+(c_i)$; assume also that $\{c_i\}$ has no accumulation point on α_0 (such a sequence can always be found as α_0 has no future endpoint). By condition (b) of Lemma 1, there must exist, for each i, a null geodesic λ_i running from c_i to some point on $\mathcal{J}^+$. Moreover, each λ_i shall be a generator of the achronal boundary $\dot{I}^+(c_i, \overline{M})$ and shall be future-complete as it reaches $\mathcal{J}^+$.

Let us now extend each of the λ_i maximally into the past. Since each λ_i passes through $c_i \in \alpha_0 \cap D^+(S)$, each of the extended λ_i must intersect S at some point $a_i \in A$. As the set A is compact, the sequence $\{a_i\}$ must have an accumulation point $a \in A$. Therefore, by Lemma 6.2.1 of Ref. [16], through a there is a non-spacelike curve λ which is future-inextendible in M and which is a limit curve of the sequence $\{\lambda_i\}$. Since all the λ_i are null geodesics, λ must be a null geodesic as well. Note also that λ must be contained in $\overline{P_0}$; otherwise λ would have to intersect the curve α_0 at some point that would be an accumulation point of the sequence $\{c_i\}$, which is impossible as $\{c_i\}$ has no accumulation point on α_0. Moreover, as P_0 is a minimal element of $\widehat{P}$, we must have $P_0 = I^-(\lambda)$. Since λ intersects the surface S, it cannot be a generator of $H^+(S)$. Thus, as $\lambda \subset \overline{P_0} \subset \overline{D(S)}$, we will have $\lambda \subset \text{int} D(S)$. Since $\lambda \subset \overline{P_0} \subset \overline{I^-(r)}$, and $r \in M$, λ cannot have a future endpoint on $\mathcal{J}^+$. In addition, as $r \in J^-(p, \overline{M})$, λ must be contained in the closure of $I^-(p, \overline{M})$. Therefore, as $p \in \mathcal{J}^+$, by condition (iii) of Theorem 1, λ must be incomplete in the future.

By condition (iv) of Theorem 1, λ must terminate in the future at a generalized strong curvature singularity. Thus the sequence $\{\lambda_i\}$ must satisfy at least one of conditions (i) and (ii) of Definition 2. Since $\lambda \subset \overline{P_0}$, and each of the λ_i must leave $\overline{P_0}$ as it has a future endpoint on $\mathcal{J}^+$, we will have $I^-(\lambda) \neq I^-(\lambda_i)$ for each i. This means that condition (ii) of Definition 2 cannot hold for the $\{\lambda_i\}$, hence λ cannot be of type A. Suppose that λ were of type B. Then, according to Definition 3, there would exist a geodesic $\widetilde{\lambda} \in \{\lambda_i\}$ and a point $\widetilde{q} \in \widetilde{\lambda} - I^-(\lambda)$ such that the expansion θ of a future-directed congruence of null geodesics outgoing from $\widetilde{q}$ and containing $\widetilde{\lambda}$ would be negative somewhere on $\widetilde{\lambda}$. As $\widetilde{\lambda}$ is future-complete, by condition (i) of our theorem and Proposition 4.4.4 of Ref. [16], there would thus exist some point $x \in J^+(\widetilde{q})$ conjugate to $\widetilde{q}$ along $\widetilde{\lambda}$. Consequently, by Proposition 4.5.12 of Ref. [16], there would also exist a timelike curve from $\widetilde{q}$ to some point $y \in \widetilde{\lambda} \cap J^+(x)$. But this is impossible because from the above construction it follows that the points $\widetilde{q}$ and y would have to lie on the achronal segment of $\widetilde{\lambda}$ contained in the boundary $\dot{I}^+(\alpha_0 \cap \widetilde{\lambda})$. This contradiction shows that the geodesic λ cannot be of type B. Thus, as λ is neither of type A nor B, it must be of type C, which completes the proof. $\square$

5 Concluding Remarks

In Theorem 1 we have demonstrated that the only possible naked singularities of strong curvature are those corresponding to null geodesics of type C, according to the classification proposed in this paper. As a consequence, one only needs to consider the stability of type C singularities as relevant to the cosmic censorship problem, which is a significant restriction. One argument showing that these singularities have a tendency to be unstable under generic perturbations was given in Ref. [15]. We may conclude that a further study of the properties of type C singularities might bring us closer to formulating the genericity and stability criterium which is needed, as mentioned in the Introduction, to arrive at a satisfactory statement of the CCH.

As we mentioned above, naked strong curvature singularities are present in certain exact solutions of the Einstein equations — solutions that represent gravitational collapse. In view of our result, it would therefore be of interest to verify whether in those solutions, the naked singularity is also associated in every case with the existence of type C geodesics. It should be stressed that this does not follow directly from Theorem 1, since it is not clear whether the solutions in question necessarily satisfy all the assumptions of that theorem.

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References

[1] R. Penrose, *Riv. Nuovo Cimento* **1**, 252 (1969).

[2] R. Penrose, in *Theoretical Principles in Astrophysics and Relativity*, eds. N. R. Lebovitz *et al.* (Univ. of Chicago Press, Chicago, 1978), pp. 217–243.

[3] F. J. Tipler, C. J. S. Clarke and G. F. R. Ellis, in *General Relativity and Gravitation*, ed. A. Held (Plenum, New York, 1980), Vol. 2, pp. 97–206.

[4] A. Królak, *Class. Quantum Grav.* **3**, 267 (1986).

[5] G. F. R. Ellis and B. G. Schmidt, *Gen. Rel. Grav.* **8**, 915 (1977).

[6] F. J. Tipler, *Phys. Lett.* **A64**, 8 (1977).

[7] A. Królak, *J. Math. Phys.* **28**, 138 (1987); *J. Math. Phys.* **33**, 701 (1992).

[8] S. S. Deshingkar, P. S. Joshi and I. H. Dwivedi, *Phys. Rev.* **D59**, 044018 (1999).

[9] P. S. Joshi, *Global Aspects in Gravitation and Cosmology* (Clarendon Press, Oxford, 1993).

[10] A. Królak, A. Czyrka, J. Gaber and W. Rudnicki, in *Proceedings of the Cornelius Lanczos International Centenary Conference*, eds. J. D. Brown *et al.* (SIAM, Philadelphia, 1994), pp. 518–520.

[11] P. S. Joshi and A. Królak, *Class. Quantum Grav.* **13**, 3069 (1996).

[12] C. S. Unnikrishnan, *Gen. Rel. Grav.* **26**, 655 (1994); *Phys. Rev.* **D53**, R580 (1996).

[13] W. B. Bonnor, *Commun. Math. Phys.* **51**, 191 (1976).

[14] W. Rudnicki, *Acta Cosm.* **18**, 77 (1992).

[15] W. Kondracki, A. Królak and W. Rudnicki, *Math. Proc. Camb. Phil. Soc.* **114**, 379 (1993).

[16] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, Cambridge, 1973).

[17] C. J. S. Clarke and A. Królak, *J. Geom. Phys.* **2**, 127 (1985).

[18] F. J. Tipler, *Int. J. Theor. Phys.* **25**, 617 (1986).

[19] C.-M. Claudel, *LANL archive preprint* gr-qc/0005031.

[20] W. Rudnicki, *Phys. Lett.* **A208**, 53 (1995).

[21] R. P. A. C. Newman and P. S. Joshi, *Ann. Phys. (N.Y.)* **182**, 112 (1988).

[22] F. J. Tipler, *Phys. Rev. Lett.* **37**, 879 (1976).

[23] A. Królak and W. Rudnicki, *Gen. Rel. Grav.* **25**, 423 (1993).